BRST Symmetry

global symmetry of the quantum action of YM

Lorentz gauge :

$$\mathcal{L}_{qu} = \underbrace{-\frac{1}{4g^2}(F_{\mu\nu}^a)^2}_{\mathcal{L}_{YM}} \underbrace{-\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2}_{\mathcal{L}_{fix}} \underbrace{-(\partial^\mu b_a)(D_\mu c)^a}_{\mathcal{L}_{ghost}}$$

$$F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f_{bc}{}^a A_\mu^b A_\nu^c$$

$$(D_\mu c)^a \equiv \partial_\mu c^a + f_{bc}{}^a A_\mu^b c^c$$

or

$$F_{\mu\nu}^a T_a = [D_\mu, D_\nu]$$

$$(D_\mu c)^a T_a = [D_\mu, c^a T_a] = (\partial_\mu c^a) T_a + [A_\mu, c^a T_a]$$

$$= [(\partial_\mu c^a) + f_{bc}{}^a A_\mu^b c^c] T_a$$

$\eta_{\mu\nu} = (-,+,+,+)$

- from representation theory of Poincare group :

  massless vectors has only 2 degree of freedom

- b_a, c^a ghosts, scalar, Grassmann variable (anti-commuting)

  b_a imaginary, c^a real to keep $\mathcal{L}_{ghost}$ real (Hermitian)

- b_a, c^a has dof (-2), cancel longitudinal and temporal dof of A_μ^a

• mass dimension: $[A_\mu^a]=1$, $[x^\mu]=-1$, $[b]+[c]=2$

choose $[b]=[c]=1$

in some literature $[b]=2$ $[c]=0$

Becchi - Rouet - Stora / Tyutin (BRST) symmetry

$\mathcal{L}_{YM}$ is invariant under gauge / local transformation

$$\delta A_\mu^a = (D_\mu \lambda)^a \equiv \partial_\mu \lambda^a + f_{bc}{}^a A_\mu^b \lambda^c$$

or $\delta A_\mu = [D_\mu, \lambda]$

$\mathcal{L}_{qu}$ is not gauge invariant (due to $\mathcal{L}_{fix}$)

but there is a global symmetry left (BRST)

replace $\lambda^a(x)$ with $c^a(x)\Lambda$,

Λ: constant, anti-commuting, imaginary

s.t. $\lambda^a(x)^\dagger = \Lambda^\dagger c^a(x)^\dagger = -\Lambda c^a(x) = c^a(x)\Lambda = \lambda^a(x)$

$[\Lambda]=-1$ $([\lambda^a]=0])$ ghost # of $\Lambda = -1$

(ghost # of $b, c = 1$)

• gauge fields

$$\delta_B A_\mu^a = (D_\mu c)^a \Lambda$$

• matter fields

$$\delta_B \psi^i = -(T_a)^i{}_j \psi^j c^a \Lambda$$

clearly $\delta_B \mathcal{L}_{YM} = 0$, $\delta_B \mathcal{L}_{matter} = 0$

deriving transf. rule for b_a and c^a

requirement $\delta_B \mathcal{L}_{qu} = 0 \iff \delta_B(\mathcal{L}_{fix} + \mathcal{L}_{ghost}) = 0$

$$\mathcal{L}_{fix} + \mathcal{L}_{ghost} = -\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + b_a \partial^\mu D_\mu c^a$$

$$\delta_B \mathcal{L}_{fix} = \frac{\delta \mathcal{L}_{fix}}{\delta A_\mu^a} \delta_B A_\mu^a$$

contains A_μ^a and c^a only

$(\delta_B A_\mu^a = (D_\mu c)^a \Lambda)$

$$\delta \mathcal{L}_{ghost} = (\delta_B b_a) \partial^\mu D_\mu c^a + \underbrace{b_a \delta_B (\partial^\mu D_\mu c^a)}_{\text{contains } b \text{ (vanish on itself)}}$$

these 2 has to cancel

explicitly $\delta_B \mathcal{L}_{fix} = -\frac{1}{\xi}(\partial^\mu A_\mu^a)(\partial^\mu \delta_B A_\mu^a) = -\frac{1}{\xi}(\partial^\mu A_\mu^a)\partial^\mu (D_\mu c)^a \Lambda$

$$\Rightarrow \delta_B b_a = -\frac{1}{\xi}(\partial^\mu A_\mu^a) g_{ab} \Lambda$$

Killing form, will be taken to be δ_{ab} from now on

now we need to make $\delta_B(\partial^\mu D_\mu c^a) = 0$ to make $\delta_B(\mathcal{L}_{fix} + \mathcal{L}_{gh}) = 0$

we make a stronger statement $\delta_B D_\mu c^a = 0$

$$0 = \delta_B D_\mu c = \delta_B [D_\mu, c] = [\delta_B A_\mu, c] + [D_\mu, \delta_B c]$$

$$\Rightarrow (D_\mu \delta_B c) + [\delta_B A_\mu, c] = 0$$

$$[\delta_B A_\mu, c] = [(D_\mu c)\Lambda, c] = [[D_\mu, c\Lambda], c]$$

$$[[D_\mu, c\Lambda], c] + [[c\Lambda, c], D_\mu] + [[c, D_\mu], c\Lambda] = 0$$

$$\Rightarrow [D_\mu, [c\Lambda, c]] = [[D_\mu, c\Lambda], c] - [[D_\mu, c], c\Lambda]$$

$$\Rightarrow [D_\mu, [c\Lambda, c]] = 2[[D_\mu, c\Lambda], c]$$

$$\Rightarrow [\delta_B A_\mu, c] = \frac{1}{2}[D_\mu, [c\Lambda, c]] = \frac{1}{2}[D_\mu, f_{bc}{}^a c^b \Lambda c^c T_a]$$

plug the result in $(D_\mu \delta_B c) + [\delta_B A_\mu, c] = 0$

$$\Rightarrow D_\mu \delta_B c + \frac{1}{2}[D_\mu, f_{bc}{}^a c^b \Lambda c^c T_a] = 0$$

in component form $(D_\mu \delta_B c)^a + \frac{1}{2} D_\mu (f_{bc}{}^a c^b \Lambda c^c) = 0$ ✱

this equation contains terms with A or without A, they must vanish seperately

$$\partial_\mu \delta_B c^a - f_{bc}{}^a (\partial_\mu c^b) c^c \Lambda = 0$$

$$\Rightarrow \delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda$$

$\boxed{\text{Q: plug back in ✱ to check this is the right solution}}$

We found a partial solution of ✱ $\qquad \partial_\mu \tilde{c}^a = 0$

the general sol. $\delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda + \tilde{c}^a$

no such $\tilde{c}^a$ in terms of fields and their derivatives

so $\quad \delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda \qquad$ (transf. rule of c^a)

note: we only need $\delta_B \partial^\mu (D_\mu c)^a$ vanish

similarly A_μ^a independent constraint is

$$\partial^\mu \partial_\mu \delta_B c^a + \partial^\mu (f_{bc}{}^a (\partial_\mu c^b) \Lambda c^c) = 0$$

Solution is the perticular sol. + general sol of $\partial^\mu \partial_\mu \delta_B c^a = 0$

only 0 sol for $\partial^\mu \partial_\mu \delta_B c^a = 0$

$$\Rightarrow \quad \delta_B \partial^\mu (D_\mu c)^a = 0 \iff \delta_B (D_\mu c)^a = 0$$

Summary: BRST symmetry in Lorentz gauge

$$\mathcal{L}_{qu} = \mathcal{L}_{YM} + \mathcal{L}_{fix} + \mathcal{L}_{ghost} = -\frac{1}{4g^2}(F_{\mu\nu}^a)^2 - \frac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + b_a \partial^\mu D_\mu c^a$$

is invariant under BRST symmetry

$$\delta_B A_\mu^a = (D_\mu c)^a \Lambda, \quad \delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda, \quad \delta_B b_a = -\frac{1}{\xi} \partial^\mu A_{\mu,a} \Lambda$$

BRST symmetry in other gauges

$$\mathcal{L}_{fix} = -\frac{1}{2} \gamma_{ab} f^a f^b \qquad \mathcal{L}_{ghost} = b_a(x) \frac{\delta f^a}{\delta \lambda^b(x)} c^b(x)$$

notice: $\dfrac{\delta f^a}{\delta \lambda^b(x)} c^b(x) = \delta_B f^a / \Lambda$, if $\delta_B A_\mu^a = (D_\mu c)^a \Lambda$

means remove Λ in the expression

$$\delta_B(\mathcal{L}_{fix} + \mathcal{L}_{ghost}) = \delta_B\left(-\frac{1}{2}\gamma_{ab} f^a f^b\right) + (\delta_B b_a) \delta_B f^a/\Lambda + b_a \delta_B^2 f^a/\Lambda$$

then $\delta_B(\mathcal{L}_{fix} + \mathcal{L}_{ghost}) = 0$

$$\Rightarrow \delta_B b_a = -\gamma_{ab} f^b \Lambda$$

$$\delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda$$

$\delta_B c^a$ is defined such that $\delta_B^2 A_\mu^a = 0$

note : • $\delta_B f^a / \Lambda$ means in the expression of $\delta_B f^a$, firstly move Λ to the far right, then remove (to make signs consistent)

• We also introduce operator s, s.t. $sF = \delta_B F / \Lambda$

ex: $sA_\mu^a = (D_\mu C)^a$ $\quad sC^a = \frac{1}{2} f_{bc}{}^a C^b C^c$

$sb_a = -\frac{1}{3} \partial^\mu A_{\mu,a}$

• all BRST transf. rules so far are for infinitesimal transf. however, linear term in Λ is enough because $\Lambda^2 = 0$

• in terms of forms. $A = A_\mu^a T_a dx^\mu$ $\quad C = C^a T_a$

assuming ghost anti-commutes with dx^μ, then

$sA = dC + \{A, C\}$, $\quad sC = CC$

because C can be viewed as 1-form $C = C^a{}_b d\lambda^b$ where λ^b are group coordinates

Nilpotency and auxiliary field

the BRST transf. laws of A_μ^a, c^a are nilpotent.
i.e. $\delta_B^2 = 0$

- $\delta_B A_\mu^a = (D_\mu c)^a$, $\delta_B c^a$ is chosen s.t $\delta_B (D_\mu c)^a = 0$
  hence $\delta_B^2 A_\mu^a = 0$

- $\delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda$

$$\delta_B(\Lambda_1)\delta_B(\Lambda_2)\, c^a = \delta_B(\Lambda_1) \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda_2$$

$$= f_{bc}{}^a (\delta_B(\Lambda_1) c^b) c^c \Lambda_2 = f_{bc}{}^a \left(\frac{1}{2} f_{pq}{}^b c^p c^q \Lambda_1\right) c^c \Lambda_2$$

$$= \frac{1}{2} f_{bc}{}^a f_{pq}{}^b \underline{c^p c^q c^c} \Lambda_2 \Lambda_1$$ totally anti-symmetric in p, q, c

recall the Jacobi identity: $f_{bc}{}^a f_{pq}{}^b + (c, p, q \text{ cyclic}) = 0$

$$\Rightarrow \delta_B(\Lambda_1)\delta_B(\Lambda_2)\, c^a = 0$$

BRST transf. law for b_a is a bit different though.

- $\delta_B b_a = -\frac{1}{3} \partial^\mu A_{\mu,a} \Lambda$

$$\delta_B(\Lambda_2)\delta_B(\Lambda_1) = \delta_B(\Lambda_2)\left(-\frac{1}{3}\partial^\mu A_{\mu,a}\right)\Lambda_1 = -\frac{1}{3}\partial^\mu (D_\mu c)_a \Lambda_2 \Lambda_1$$

$\delta_B^2 b_a$ is 0 only on-shell ($\partial^\mu D_\mu c^a = 0$ is e.o.m $\frac{\delta S}{\delta b_a} = 0$)

to make the whole BRST symmetry nilpotent off-shell, introduce <u>auxiliary field d_a</u>

$$\mathcal{L}_{fix,aux} = \frac{1}{2}\xi(d_a)^2 + d_a \partial^\mu A_\mu^a$$

the whole quantum action is BRST invariant if

$$\delta_B b_a = d_a \Lambda \qquad \delta_B d_a = 0$$

integrate over d_a we have $d_a = -\frac{1}{\xi}\partial^\mu A_{\mu,a}$

and $\mathcal{L}_{fix,aux}$ becomes $\mathcal{L}_{fix} = -\frac{1}{2\xi}(\partial^\mu A_\mu^a)^2$

note: d_a makes BRST symm. close

Summary: the BRST transf. rules with aux. field

$$\begin{cases} \delta_B A_\mu^a = (D_\mu c)^a \Lambda \\ \delta_B c^a = \frac{1}{2} f_{bc}{}^a c^b c^c \Lambda \\ \delta_B b_a = d_a \Lambda \\ \delta_B d_a = 0 \end{cases}$$

the transf. rules are nilpotent $\delta_B^2 = 0$

the complete quantum action is

$$\mathcal{L}_{qu} = \mathcal{L}_{YM} + \delta_B\left(b_a\left(f^a + \frac{1}{2}\xi d^a\right)\right)/\Lambda$$

$$= \mathcal{L}_{YM} + s\left(b_a\left(f^a + \frac{1}{2}\xi d^a\right)\right)$$

where f^a is the gauge fixing term

$\delta_B \mathcal{L}_{YM} = 0$ (gauge inv.)

the 2nd term is BRST inv. because of the nilpotency

BRST charge and physical states

$$\mathcal{L}_{qu} = \mathcal{L}_{classical} + s\Psi$$

$$\Psi = b_a \left(f^a + \frac{1}{2}\xi d^a\right) \longrightarrow \text{gauge dependent}$$

BRST charge Q_B :

$$s\mathcal{O} = [Q_B, \mathcal{O}\}$$

$[-,-\}$: means commutator or anti-commutator depending on statistics

Q_B : fermionic . $Q_B^2 = 0$

S-matrix should be independent of $s\Psi$ (independent of gauge)

$\Rightarrow$ $\langle\alpha|\beta\rangle$ unchanged when $\Psi \longrightarrow \Psi'$

$\Rightarrow$ $\langle\alpha|s\Psi|\beta\rangle = \langle\alpha|\{Q_B, \Psi\}|\beta\rangle = 0$ for all Ψ

$\Rightarrow$ $\langle\alpha|Q_B = Q_B|\beta\rangle = 0$

- physical states must be BRST-closed : $Q_B|\text{phys}\rangle = 0$

on the other hand, consider $|\beta\rangle$ and $|\beta\rangle + Q_B|\gamma\rangle$ for any physical states $\langle\alpha|$

$$\langle\alpha|\beta\rangle = \langle\alpha|(|\beta\rangle + Q_B|\gamma\rangle) \qquad \forall \langle\alpha|$$

$\Rightarrow$ $|\beta\rangle$, $|\beta\rangle + Q_B|\gamma\rangle$ are physically equivalent

- states differ by a BRST-exact term $Q|\cdots\rangle$ are equivalent

Summary: physical states are BRST-closed modulo BRST-exact
i.e. in BRST-cohomology

- when acting on no ghost fields (A_μ^a, ψ^i ...), BRST transf. are gauge transf. w/ $\lambda^a_{(x)} \to c^a(x)\Lambda$

$\Rightarrow$ gauge invariant ops/states are always Q_B-closed

note: $Q_B|\alpha\rangle = 0$ & $|\beta\rangle = Q_B|\gamma\rangle \Rightarrow |\alpha\rangle \otimes |\beta\rangle = Q(|\alpha\rangle \otimes |\gamma\rangle)$
i.e. Q-closed tensor Q-exact $\to$ Q-exact

pure QED

$$f = \partial^\mu A_\mu \qquad \text{Lorentz gauge}, \quad f_{abc} = 0$$

$$\mathcal{L}_{qu} = -\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + (\partial^\mu b)\,\partial_\mu c$$

$$sA_\mu = \partial_\mu c \qquad sc = 0 \qquad sb = \frac{1}{\xi}\partial^\mu A_\mu$$

b, c decouple from A_μ, ψ, c is BRST closed.

- consider pure QED part (sA_μ, sc, sb won't increase # of fields)

$$A^\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^0}} \left[a^\mu(\vec{p})e^{ip\cdot x} + a^{\mu *}(\vec{p})e^{-ip\cdot x}\right]$$

$$c(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^0}} \left[c(\vec{p})\, e^{ip\cdot x} + c^*(\vec{p})e^{-ip\cdot x}\right]$$

$$b(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{2p^0}} \left[b(\vec{p})\, e^{ip\cdot x} + b^*(\vec{p})e^{-ip\cdot x}\right]$$

$$[Q_B, a^\mu(\vec{p})] = i p^\mu c(\vec{p}) \qquad [Q_B, a^{\mu *}(\vec{p})] = -i p^\mu c^*(\vec{p})$$

$$\{Q_B, c(\vec{p})\} = 0 \qquad \{Q_B, c^*(\vec{p})\} = 0$$

$$\{Q_B, b(\vec{p})\} = i p^\mu a_\mu(\vec{p})/\xi \qquad \{Q_B, b^*(\vec{p})\} = -i p^\mu a_\mu^*(\vec{p})/\xi$$

let $|\psi\rangle$ be a phys states, i.e. $Q_B|\psi\rangle = 0$

- $|e,\psi\rangle = e_\mu a^{\mu *}(p)|\psi\rangle$ add a photon with polarization e_μ

  $|e,\psi\rangle$ phys $\Rightarrow Q_B|e,\psi\rangle = Q_B e_\mu a^{\mu *}(p)|\psi\rangle = i e_\mu p^\mu c^*(p)|\psi\rangle = 0$

  $\Rightarrow e_\mu p^\mu = 0$ $\qquad$ e_μ : transverse to p_μ, or $e_\mu = \alpha p_\mu$ $(p^2=0)$

  on the other hand $\quad Q_B b^*(\vec{p})|\psi\rangle = -i p^\mu a_\mu^*(\vec{p})/\xi$

  $$\Rightarrow |e+\alpha p, \psi\rangle = |e,\psi\rangle + |\alpha p, \psi\rangle$$
  $$= |e,\psi\rangle + i \underbrace{Q_B \alpha b^*(\vec{p})|\psi\rangle/\xi}_{Q_B\text{-exact}}$$

  $\Rightarrow |e+\alpha p,\psi\rangle$, $|e,\psi\rangle$ belongs to the same Q-cohom.

  $\Rightarrow$ physical photon states $\quad e_\mu$: transverse to p_μ
  2 polarizations

- $Q_B b^*(p)|\psi\rangle = -i p^\mu a_p^*(\vec{p})/\xi\,|\psi\rangle$

  $Q_B b^*(p)|\psi\rangle = 0$ only when $p^\mu = 0$ $\qquad$ $b^*(p)|\psi\rangle$ is not physical except $b^*(0)|\psi\rangle$

- $$c^*(p)|\psi\rangle = -i\, Q_B \frac{e_\mu a^{\mu *}(p)}{e_\mu \cdot p^\mu}|\psi\rangle \qquad Q_B c^*(p)|\psi\rangle = 0 \text{ automatically}$$

  Q_B - exact as long as $e_\mu p^\mu \neq 0$

- $c^*(0)|\psi\rangle$, $b^*(0)|\psi\rangle$ not ruled out yet. impose ghost # $= 0$ for physical states to remove them

pure QCD

$$\mathcal{L}_{qu} = -\frac{1}{4}\left(F_{\mu\nu}^{a}\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A_\mu^a\right)^2 + \left(\partial^\mu b_a\right)\left(D_\mu c^a\right)$$

we rescale $A_\mu^a \to g A_\mu^a$, then $D_\mu = \partial_\mu + g A_\mu$

$$\delta A_\mu^a = D_\mu c^a = \partial_\mu c^a + g f_{bc}{}^{a} A_\mu^b c^c$$

if $|\psi\rangle$ is physical $Q_B|\psi\rangle = 0$

$$Q_B\, \varepsilon_\mu a^{\mu,a*}(p)|\psi\rangle = i\varepsilon_\mu p^\mu c^{a*}(p) + g f_{bc}{}^{a} \int \frac{d p'^3}{\sqrt{2p'^0}} \varepsilon_\mu a^{\mu,b*}(p-p') c^c(p')|\psi\rangle$$

In general

$Q_B\, \varepsilon_\mu a^{\mu,a*}(p)|\psi\rangle = 0$ leads to $\varepsilon_\mu = 0$

only when $g \to 0$ we ignore the 2nd term and approximate the transverse gluons as "physical" (asymptotic states, or weak interaction, not work in QCD)

note:

- BRST transf. does not commute with global part of the gauge transf. in non-Abelian case.

$\Rightarrow$ even if $Q_B\, \varepsilon_\mu a^{\mu,a,*}(p)|\psi\rangle = 0$

$$Q_B\, \varepsilon_\mu \hat{a}^{\mu,a*}(p)|\psi\rangle \neq 0 \quad \text{in general}$$

where $\hat{a}^{\mu,a*}$ is $a^{\mu,a,*}$ transformed under the global part of gauge transf.

$\Rightarrow Q_B\, \varepsilon_\mu a^{\mu,a,*}(p)|\psi\rangle \neq 0$ in general

- physical states : gauge inv. (differ from QED, because in QED, photon has no electric charge)

QED with matter

add complex scalar $\phi(x)$ to the photon $\mathcal{L}$ with charge 1

$$\mathcal{L}_{matter} = -(D_\mu \phi^*)(D^\mu \phi)$$

$$s\phi = -\phi c$$

let a^*, $\bar{a}^*$ be creation operators of the complex scalar and its anti-particle

$$[Q_B, a^*(p)] = -\int \frac{d^3p_1}{\sqrt{2p_1^0}} \int \frac{d^3p_2}{\sqrt{2p_2^0}} \delta(p-p_1-p_2)\, a^*(p_1)\, c^*(p_2)$$

also $[Q_B, a^\mu(\vec{p})] = -ip^\mu c(\vec{p})$

let $|\psi\rangle$, $Q_B|\psi\rangle = 0$

consider

$$|\phi\rangle = a^*(p)|\psi\rangle + \int \frac{d^3p_1}{\sqrt{2p_1^0}} \int \frac{d^3p_2}{\sqrt{2p_2^0}} \delta(p-p_1-p_2) \frac{i\, p_2^0 a^{0,*}(p_2)}{(p_2^0)^2} a^*(p_1)|\psi\rangle$$

$$+ \frac{1}{2} \int \frac{d^3p_1}{\sqrt{2p_1^0}} \int \frac{d^3p_2}{\sqrt{2p_2^0}} \int \frac{d^3p_3}{\sqrt{2p_3^0}} \delta(p-p_1-p_2) \frac{i\, p_3^0 a^{0,*}(p_3)}{(p_3^0)^2} \frac{i\, p_2^0 a^{0,*}(p_2)}{(p_2^0)^2} a^*(p_1)|\psi\rangle$$

$$+ \cdots$$

then $Q_B|\phi\rangle = 0$, $|\phi\rangle$ is roughly $\phi\, e^{\frac{p^0 A^0}{(p^0)^2}}|\psi\rangle$

matter with non-zero charge can be physical

- no such simple solution in QCD following similar argument before

The BRST Jacobian

we haven't show that the path-integral measure $\mathcal{D}A_\mu^a \mathcal{D}b_a \mathcal{D}c^a$ is invariant under BRST transf. To show this, we need compute the BRST Jacobian

$$\det \frac{\delta(\varphi^i(x) + \delta_B \varphi^i(x))}{\delta \varphi^j(y)} \quad \text{where } \varphi^i(x) = (b_a(x), A_\mu^a(x), c^a(x))$$

$$\det \frac{\delta(\varphi^i(x) + \delta_B \varphi^i(x))}{\delta \varphi^j(y)} = e^{\operatorname{Tr} \ln \frac{\delta(\varphi^i(x) + \delta_B \varphi^i(x))}{\delta \varphi^j(y)}}$$

$$= \delta(x-y) + \operatorname{Tr}\left(\frac{\partial \delta_B A_\mu^a(x)}{\partial A_\nu^b(y)} - \frac{\partial \delta_B c^a(x)}{\partial c^b(y)} - \frac{\partial \delta_B b_a(x)}{\partial b_b(y)} \right)$$

- we understand ∂ as δ (variation derivative) here to avoid confusing with δ_B
- Tr : sum over $a=b$, $\mu=\nu$ indices as well as integral over $x=y$, i.e. $\operatorname{Tr} \cdots = \sum_{a,b} \sum_{\mu,\nu} \int d^4x \, d^4y \, \delta(x-y) \, \delta_{ab} \, \eta_{\mu\nu} \cdots$
- "$-$" sign from the anti-commuting nature of b_a, c^a similar to "$-$" in ghost loop / fermion loop

$$J = \frac{\partial \delta_B A_\mu^a(x)}{\partial A_\nu^b(y)} - \frac{\partial \delta_B c^a(x)}{\partial c^b(y)} - \frac{\partial \delta_B b_a(x)}{\partial b_b(y)}$$

$$\frac{\partial \delta_B A_\mu^a(x)}{\partial A_\nu^b(y)} = \frac{\partial (D_\mu c(x))^a}{\partial A_\nu^b(y)} \Lambda = f_{bc}{}^a \, c^c(x) \, \delta_\mu{}^\nu \, \delta(x-y) \, \Lambda$$

formally trace over a,b. gives 0 for semi-simple Lie alg.

because f_{abc} is totally anti-symmetric, hence $f_{ac}{}^{a} = 0$

similar for $\dfrac{\partial \delta_B C^a(x)}{\partial C^b(y)}$ and $\dfrac{\partial \delta_B b_a(x)}{\partial b_b(y)}$

hence the Jacobian is unity. and the path-integral measure is BRST invariant ...

... but not too fast

there is still $\int d^4x\, d^4y\, \delta(x-y)$ which is usually ∞

in the end $\mathrm{tr}\, J = 0 \times \infty$ may not be 0

if $\mathrm{Tr}\, J \neq 0$. the Jacobian is not unity

$\Rightarrow$ the measure is not invariant under BRST transf.

$\Rightarrow$ Anomaly: $\delta_B \underbrace{\Gamma}_{\text{effective action}} = A_n$

Compute $\mathrm{Tr}\, J$

- idea: choose a regularization scheme (heat kernel reg.)

$\mathrm{Tr}\, J = A_n = \delta_B S_{loc} \neq 0$

S_{loc} is a finite local count terms

$\Gamma_{re} = \Gamma - S_{loc}$

$\Rightarrow \delta_B \Gamma_{re} = \delta_B \Gamma - \delta_B S_{loc} = 0 \Rightarrow$ BRST is anomaly free

- Consistent condition for anomalies, for a symmetry transf. λ of the action S, $\quad \mathcal{A}_n = \delta_\lambda \Gamma$

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\Gamma = \delta_{\lambda_3(\lambda_1,\lambda_2)}\Gamma$$

$$\Rightarrow \delta_{\lambda_1}\mathcal{A}_n(\lambda_2) - (1 \leftrightarrow 2) = \mathcal{A}_n(\lambda_3(\lambda_1,\lambda_2))$$

For BRST transf. $\quad \delta_{\lambda_1}\delta_{\lambda_2} = 0$

$$\Rightarrow \delta_{\lambda_1}\delta_{\lambda_2}\Gamma = 0 \quad \Rightarrow \text{ BRST anomaly is BRST-closed}$$

Theorem (O. Piguet, S.P. Sorella, Algebraic Renormalization)

If two regularization schemes ~~both~~ give BRST invariant anomalies, they differ by the BRST variation of a local counter-term

- If we show that the BRST anomaly is BRST variation of a finite local term, it is true for all reg. scheme.

heat kernel regularization

$$J = \frac{\partial \delta_B A^a_\mu(x)}{\partial A^b_\nu(y)} - \frac{\partial \delta_B c^a(x)}{\partial c^b(y)} - \frac{\partial \delta_B b_a(x)}{\partial b_b(y)}$$

$$\mathrm{Tr}\, J \longrightarrow \mathrm{Tr}\, J\, e^{R/M^2}, \qquad \mathrm{Tr} \cdots = \sum_{a,b}\sum_{\mu,\nu}\int d^4x\, d^4y\, \delta_{ab}\,\delta_\mu^{\ \nu}\,\delta^4(x-y)\cdots$$

$$R^i_{\ j} = (T^{-1})^{ik} S_{kj}, \qquad \varphi^i = \{b_a, A^a_\mu, c^a\}$$

- $\varphi^i T_{ij} \varphi^j$: any non-singular mass matrix
- S_{kj} : kinetic operator $\quad \underbrace{\frac{\partial}{\partial\varphi^i}}_{\text{act from left}} S_{qn} \underbrace{\frac{\overleftarrow{\partial}}{\partial\varphi^j}}_{\text{act from right}}$

- $\partial, \bar{\partial}$ differ by -1 when acting on b_a, c^a (anti-commuting)
- reason for this regulator, see A. Diaz, W. Troost, P. van Nieuwenhuizen, A. van Proeyen, Int J. Mod. Phys A 4 (1989) 3959
- T: nondegenerate mass term, inv. under global part of gauge symm., zero ghost #

$$\Longrightarrow \int d^4x \, \mathrm{Tr}(A_\mu A^\mu + 2bc) \quad \text{unique up to scaling}$$

$$\varphi^i = (b, A, c)$$

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \eta^{\mu\nu} & 0 \\ -1 & 0 & 0 \end{pmatrix} \delta^4(x-y)\,\delta^{ab}$$

$$T^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \eta_{\mu\nu} & 0 \\ 1 & 0 & 0 \end{pmatrix} \delta^4(x-y)\,\delta^{ab}$$

- S: differentiating the quantum action

$$S_{qu} = \int d^4x \left(-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2}(\partial^\mu A^a_\mu)^2 - (\partial^\mu b_a)(D_\mu c)^a \right) \qquad \xi = 1$$

$$S_{kl} = \begin{pmatrix} 0 & -\partial^\nu c & \partial^\rho D_\rho(A) \\ c\,\partial^\mu & R^{\mu\nu} & (\partial^\mu b) \\ -D^\rho(A)\,\partial_\rho & (\partial^\nu b) & 0 \end{pmatrix} \delta^4(x-y)$$

$$R^{\mu\nu}(x)\,\delta^4(x-y) = \frac{\partial}{\partial A_\nu(y)} D^\rho F_\rho{}^\mu(x) + \partial^\mu_x \partial^\nu_x \delta(x-y)$$

$$= \left(2F^{\mu\nu} + \eta^{\mu\nu} D^\rho(A) D_\rho(A) - D^\mu(A) D^\nu(A) + \partial^\mu \partial^\nu\right) \delta^4(x-y)$$

everything in adjoint rep: ex. $c = c_b{}^a = g f_{bc}{}^a c^c$

- $$\mathcal{J} = \begin{pmatrix} 0 & -\partial^\nu & 0 \\ 0 & -c\,\delta_\mu{}^\nu & -D_\mu(A) \\ 0 & 0 & -c \end{pmatrix} \delta^4(x-y)\,\Lambda$$

- result

$$A_n = \frac{1}{(4\pi)^2}\int d^4x \; \frac{1}{12}\,\mathrm{Tr}\,(\partial^\nu c)\Big[4A_\mu A_\nu A^\mu - 4A^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) - 4A_\nu \partial_\mu A^\mu + \partial^\mu\partial_\mu A_\nu - 3\,\partial_\nu\partial_\mu A^\mu\Big]$$

actually $A_n = \delta_B S_{loc}$

$$S_{loc} = \frac{1}{(4\pi)^2}\int d^4x \; \frac{1}{12}\,\mathrm{Tr}\,\Big[(\partial^\mu A_\mu)^2 + \frac{3}{2} A_\mu A_\nu A^\mu A^\nu - \frac{1}{2} A^2 A^2\Big]$$

there is no genuine BRST anomaly, can be removed by a local counter term

computation of $\mathrm{Tr}\,\mathcal{J}\, e^{T^{-1}S/M^2}$

recall $\phi^i = (b_a, A_\mu^a, c^a)$

① $\mathcal{J}$, T, S are 6×6 matrices $\times\, \delta^4(x-y)$. each entry is in adjoint representation

$$T^{-1}S = \begin{pmatrix} D^\rho\partial_\rho & -(\partial^\nu b) & 0 \\ c\,\partial_\rho & R_\mu{}^\nu & (\partial_\mu b) \\ 0 & -\partial^\nu c & \partial^\rho D_\rho \end{pmatrix}\delta^4(x-y)$$

rewrite this in terms of the number of derivatives

$$T^{-1}S = \left[(\partial_\alpha \mathbb{1} + Y_\alpha)\eta^{\alpha\beta}(\partial_\beta \mathbb{1} + Y_\beta) + E \right]\delta^4(x-y)$$

where $\mathbb{1}$, Y_α, E are 6×6 matrices without any free derivatives.

$$\underline{1} = \begin{pmatrix} \delta^a{}_b & & \\ & \delta_a{}^b \delta^\mu{}_\nu & \\ & & \delta_a{}^b \end{pmatrix}$$

$$Y_\alpha = \frac{1}{2}\begin{pmatrix} A_\alpha & 0 & 0 \\ c\,\delta^\mu{}_\alpha & (A_\alpha \delta^\mu{}_\nu - \frac{1}{2}A^\mu \eta_{\nu\alpha} - \frac{1}{2}A_\nu \delta^\mu{}_\alpha) & 0 \\ 0 & -c\,\eta_{\alpha\nu} & A_\alpha \end{pmatrix}$$

$$E = \begin{pmatrix} -\frac{1}{2}\partial^\alpha A_\alpha & -\partial^\nu b & 0 \\ -\frac{1}{2}\partial^\mu c & \left\{ \begin{matrix} \frac{3}{2}(\partial^\mu A_\nu - \partial_\nu A^\mu) + \eta^\mu{}_\nu A^\rho A_\rho \\ + A^\mu A_\nu - 2A_\nu A^\mu \end{matrix} \right\} & \partial^\mu b \\ 0 & -\frac{1}{2}\partial_\nu c & \frac{1}{2}\partial^\alpha A_\alpha \end{pmatrix}$$

$$- \begin{pmatrix} \frac{1}{4}A^\alpha A_\alpha & 0 & 0 \\ \frac{1}{4}cA^\mu - \frac{3}{4}A^\mu c & \frac{1}{4}A^\alpha A_\alpha \delta^\mu{}_\nu + \frac{1}{2}A^\mu A_\nu - A_\nu A^\mu & 0 \\ -c^2 & -\frac{1}{4}A_\nu c + \frac{3}{4}c A_\nu & \frac{1}{4}A^\alpha A_\alpha \end{pmatrix}$$

note: $\partial^\alpha A_\alpha$, $\partial^\mu b$ in E understood as $(\partial^\alpha A_\alpha)$, $(\partial^\mu b)$

② symmetrizing J : to remove free derivatives in J

logic: $\partial\phi + (\partial\phi)^T = \partial\phi + \underset{\substack{\uparrow \\ \text{Hermitian}}}{\phi^T}\partial^T = \partial\phi - \phi\partial = (\partial\phi)$

In practice, have to keep track of signs and fermi/bose (anti) commutation properties

easier to start from the mass term $\phi^i T_{ij} \phi^j$

$$\delta_B(\phi^i T_{ij}\phi^j) = \phi^i(T_{ij}\delta_B\phi^j) + (\delta_B\phi^j)T_{jk}\phi^k$$

Symmetrized Jacobian:

$$j_{sym}\delta^4(x-y) = \underbrace{\partial\delta_B\phi^i/\partial\phi^j}_{J} + \underbrace{(T^{-1})^{ik}\left(\frac{\partial}{\partial\phi^k}\delta_B\phi^l\right)T_{lj}}_{T^{-1}J^T T}$$

$$\mathrm{Tr}\, T^{-1}J^T T e^{T^{-1}S/M^2} = \mathrm{Tr}\, T e^{T^{-1}S/M^2} T^{-1}J^T$$

$$= \mathrm{Tr}\, e^{ST^{-1}/M^2} J^T, \qquad \mathrm{Tr}\, A = \mathrm{Tr}\, A^T$$

recall T, S symmetric, $\quad T^{-1}S = (T^{-1}S)^T = ST^{-1}$

$$\Rightarrow \mathrm{Tr}\, J e^{ST^{-1}/M^2} = \mathrm{Tr}\, T^{-1}J^T T e^{T^{-1}S/M^2}$$

$$A_n = \mathrm{Tr}\, J e^{T^{-1}S/M^2} = \frac{1}{2}\mathrm{Tr}\, j_{sym}\delta^4(x-y) e^{T^{-1}S/M^2}$$

$$j_{sym} = \underbrace{\begin{pmatrix} c & A^\nu & 0 \\ 0 & 0 & -A_\mu \\ 0 & 0 & -c \end{pmatrix}}_{\text{traceless}} \Lambda \qquad \text{with no free derivative}$$

(3) compute $A_n = \mathrm{Tr}\, J\, e^{T^{-1}S/M^2}$

$$e^{T^{-1}S/M^2} = e^{(D^\alpha D_\alpha + E)/M^2}\,\delta^4(x-y)$$

$$= \langle y|\, e^{(D^\alpha D_\alpha + E)/M^2}\,|x\rangle$$

use: $\langle y|x\rangle = \delta^4(x-y)$, with $D_\alpha = \partial_\alpha + Y_\alpha$

$$A_n = \int d^4x \int d^4y\ \mathrm{str}\,\langle x|\, j_{sym}(x)\,|y\rangle\langle y|\, e^{(D^\alpha D_\alpha + E)/M^2}\,|x\rangle$$

$$= \frac{1}{2}\int d^4x \int d^4y\ \mathrm{str}\ j_{sym}(x)\,\underbrace{\delta^4(x-y)}_{\langle x|y\rangle = \int \frac{d^4k}{(2\pi)^4}\langle x|k\rangle\langle k|y\rangle}\,\langle y|\, e^{(D^\alpha D_\alpha + E)/M^2}\,|x\rangle$$

$$= \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\int d^4y\ \mathrm{str}\ j_{sym}\,\langle x|k\rangle\langle k|y\rangle\langle y|\, e^{(D^\alpha D_\alpha + E)/M^2}\,|x\rangle$$

$$= \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\ \mathrm{str}\ j_{sym}\,\underline{\langle x|k\rangle}\,\underbrace{\langle k|\, e^{(D^\alpha D_\alpha + E)/M^2}\,|x\rangle}_{e^{(D^\alpha D_\alpha'' + E)/M^2}\langle k|x\rangle}$$

$$= \frac{1}{2}\int d^4x \int \frac{d^4k}{(2\pi)^4}\ \mathrm{str}\ j_{sym}\ \underline{e^{-ikx}}\ \underline{e^{(D^\alpha D_\alpha + E)/M^2}\, e^{ikx}}$$

$$= \int d^4x\ \mathrm{str}\ j_{sym}(x)\, h(x,x)$$

$$h(x,y) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx}\, e^{(\underline{D^\alpha D_\alpha + \overset{R(x)}{E}})/M^2}\, e^{ikx}$$

note: str is the supertrace over the 6×6 matrices and gauge indices

④ heat kernel

$$h(x,x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} e^{R(x)/M^2} e^{ikx}$$

$\partial_\alpha \to \partial_\alpha + ik_\alpha$

$$= \int \frac{d^4k}{(2\pi)^4} e^{(-k^2+2ik^\alpha D_\alpha + R(x))/M^2}$$

rescaling $K = k/M$

$$h(x,x) = M^4 \int \frac{d^4K}{(2\pi)^4} e^{-K^2} \exp\left[\frac{2iK^\alpha D_\alpha}{M} + \frac{R(x)}{M^2}\right]$$

expand in terms of M, we keep only terms with positive power of M, as $M\to\infty$ in the end (regularization)

leading term $\int d^4x \, \underline{\mathrm{str}\, j_{sym}} \, M^4 \int \frac{d^4k}{(2\pi)^4} e^{-k^2} = 0$

↳ j_{sym} along is traceless

NL , NNL ...

only non zero term is $O(M^0)$

$$a_2 = M^4 \int \frac{d^4K}{(2\pi)^4} e^{-K^2}\Big[\frac{1}{2!}\left(\frac{D^\alpha D_\alpha + E}{M^2}\right)^2 + \frac{1}{3!}\Big\{\frac{D^\beta D_\beta + E}{M^2}\frac{2iK^\alpha D_\alpha}{M}\frac{2iK^\gamma D_\gamma}{M}$$
$$+ \frac{2iK^\alpha D_\alpha}{M}\frac{(D^\beta D_\beta + E)}{M^2}\frac{2iK^\gamma D_\gamma}{M} + \frac{2iK^\alpha D_\alpha}{M}\frac{2iK^\gamma D_\gamma}{M}\frac{D^\beta D_\beta + E}{M^2}\Big\}$$
$$+ \frac{1}{4!}\frac{(2iK^\alpha D_\alpha)^4}{M^4}\Big]$$

Anti-BRST symmetry

idea: exchange the role of b_a and c^a

$$\underbrace{\delta_{\bar{B}}}_{\text{anti-BRST}} A_\mu^a = \underbrace{(D_\mu b)^a}_{\text{anti-Hermitian}} \underbrace{\zeta}_{\text{anti-commuting, real parameter}}$$

$[\zeta] = -1$, ghost # +1

Nilpotency: $\delta_{\bar{B}}^2 A_\mu^a = 0$

$$\Rightarrow \delta_{\bar{B}} b^a = \frac{1}{2} f_{bc}{}^a b^b b^c \zeta$$

transformation rule for $\delta_{\bar{B}} c^a$ $\delta_{\bar{B}} d^a$

assume same BRST transf. rule as before

ansatz: constrain by dim, ghost #, Lorentz inv. reality

$$\delta_{\bar{B}} c \sim d\zeta + \partial\cdot A\zeta + bc\zeta$$

$$\delta_{\bar{B}} d \sim bd\zeta + b\,\partial\cdot A\zeta$$

also require that BRST, anti-BRST both nilpotent and commute with each other:

$$\delta_B(\Lambda_1)\delta_B(\Lambda_2) = \delta_{\bar{B}}(\Lambda_1)\delta_{\bar{B}}(\Lambda_2) = [\delta_B(\Lambda), \delta_{\bar{B}}(\zeta)] = 0$$

$$\Rightarrow \begin{cases} \delta_{\bar{B}} c^a = -d^a\zeta + f_{bc}{}^a b^b c^c \zeta \\ \delta_{\bar{B}} d^a = -f_{bc}{}^a b^b d^c \zeta \end{cases}$$

- use $\bar{s}$ to denote $\delta_{\bar{B}} \cdots / \zeta$.

quantum action invariant under δ_B and $\delta_{\bar{B}}$

$$\mathcal{L}_{qu} = \mathcal{L}_{YM} + s\Big(\underbrace{b_a\big(F^a + \tfrac{1}{2}\xi d^a\big)}_{\hookrightarrow\ \Psi}\Big)$$

$\mathcal{L}_{YM}$ is both BRST / anti-BRST invariant

- if $s\Psi = s\bar{s}\underbrace{X}_{[X]=2,\ \text{ghost}\ \#\ 0}$, $\mathcal{L}_{qu}$ will be both BRST/anti-BRST inv.

Lorentz inv. $\Rightarrow$ $b_a\big(F^a + \frac{1}{2}\xi d^a\big) = \bar{s}\big(\alpha (A_\mu^a)^2 + \beta\, b_a c^a\big)$

no solution if $F^a \propto \partial^\mu A_\mu^a$

- weaker requirement

$$s\Psi = \bar{s}\,Y \quad , \qquad \text{no solution if } F^a \propto \partial^\mu A_\mu^a$$

- simple gauge-fixing term won't work

$$F^a = \partial^\mu A_\mu^a - \tfrac{1}{2}\xi f_{bc}{}^a b^b c^c$$

Curci-Ferrari model

$$\begin{aligned}
\mathcal{L}_{qu} &= \mathcal{L}_{cl} + s\bar{s}\big(-(A_\mu^a)^2 - \tfrac{1}{2}\xi\, b_a c^a\big) \\
&= \mathcal{L}_{cl} + s\big(b_a\big(F^a + \tfrac{1}{2}\xi d^a\big)\big) \\
&= \mathcal{L}_{cl} - \tfrac{1}{2\xi}(\partial^\mu A_\mu^a)^2 + \tfrac{1}{2} b_a(\partial^\mu D_\mu + D_\mu \partial^\mu)c^a \\
&\quad + \tfrac{1}{8}\xi\,(f_{bc}{}^a b^b c^c)^2 + \tfrac{1}{2}\xi\big(d^a + \tfrac{1}{\xi}\partial^\mu A_\mu^a - \tfrac{1}{2} f_{bc}{}^a b^b c^c\big)^2
\end{aligned}$$

renormalizable but non-unitary.

Can have BRST/anti-BRST inv mass term $-\frac{1}{2}m^2 (A_\mu^a)^2 - \xi m^2 b_a c^a$